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MASS TRANSFER IN A DIFFUSION TRAIL OF A DROP IN STOKES FLOW

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Diffusion of a substance in the region behind a spherical drop is investigated in a Stokes flow for large values of the Péclet number. The method of matching asymptotic expansions is used to obtain the distribution of concentration in the diffusive trail of the drop, and to determine the local diffusive flux towards the stern part of its surface. It is shown that behind the drop the concentration increases linearly along the symmetry axis of the flow with the distance from the rear stagnation point, and increases with the increasing ratio β of the viscosity of the drop and the surrounding fluid. The local Sherwood number attains its minimum at the rear stagnation point, and is equal to $[6\pi^{-1}(\beta + 1)P^{-1}]^{1/2}$ (P is the Péclet number). The thickness of the diffusive boundary layer is of the order of $P^{1/2}$.

The solution of the problem of mass transfer in a drop at large Péclet numbers obtained by the method of diffusive boundary layers, becomes unsuitable in the region of the diffusive trail including the neighborhood of the rear stagnation point [1]. Within this region the method leads to a boundary layer of infinitely increasing thickness. Nevertheless, the study of the distribution of concentration within the region constitutes an important stage in the analysis of the diffusive interaction between the particles or drops. It also makes possible the removal of the singularity from the rear stagnation point.

A similar problem was investigated in [2 — 4] for a rigid sphere. The mobility of the drop surface determines the disparities in the flow pattern and distribution of concentration of the drop and the rigid sphere.

1. Formulation of the problem. The distribution of concentration over the whole region outside the drop undergoing a translational motion with velocity U at small values of the Reynolds number is given, in the assumption that the substance is fully absorbed at the drop surface and has constant concentration away from the drop, by the solution of the following boundary value problem:

$$v_r \frac{\partial c}{\partial r} + \frac{v_\theta}{r} \frac{\partial c}{\partial \theta} = e^2 \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial c}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial c}{\partial \theta} \right\} \tag{1.1}$$

$$v_r = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}, \quad e^{-2} = P = \frac{aU}{D}$$

$$\psi = \frac{1}{4}(r-1)\left(2r - \frac{\beta}{\beta+1} - \frac{\beta}{\beta+1}\frac{1}{r}\right)\sin^2\theta$$

$$r = 1, \quad c = 0; \quad r = \infty, \quad c = 1 \tag{1.2}$$

$$\theta = \pi, \quad \partial c / \partial \theta = 0; \quad \theta = 0, \quad \partial c / \partial \theta = 0$$

Here we use a spherical coordinate system with the origin at the center of the drop (the angle θ is counted from the direction of the flow at infinity, and the radial coordinate is referred to the drop radius a) and the axial symmetry of the flow is taken into account. The concentration at infinity is used as the unit concentration.

We note that the last two conditions in (1.2) follow from the symmetry of the problem. They play the necessary part of the supplementary conditions when the method of matching asymptotic expansions is used and the initial equation (1.1) is replaced in the corresponding regions by a parabolic type equation.

We consider the case of large Péclet numbers, i.e. $e \ll 1$. The solution of the problem (1.1), (1.2) in this case is known [1] everywhere outside the drop except in the region of the diffusive trail $W = \{0 \leqslant \theta < e,\ r \geqslant 1\}$. The aim of this work is to construct a solution for this region.

Expansion of the unknown function c into a series in terms of the small parameter e does not produce a solution which would be uniformly applicable over the whole region W. For this reason we must partition this region into several subregions (see Fig. 1) where numbers denote the corresponding subregions W^i $(i = 1, 2, 3, 4)$ and d denotes the region of the diffusive boundary layer), construct an asymptotic solution in each of these subregions and match the asymptotics on the specified boundaries.

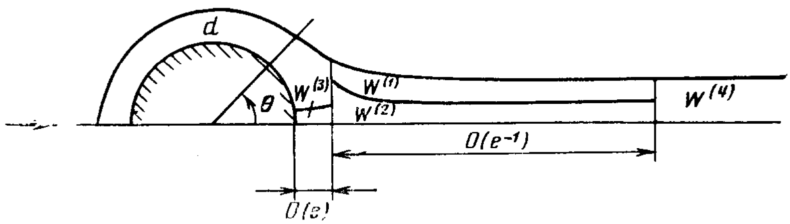

Fig. 1

2. Distribution of concentration in the region of diffusive trail of the drop. Let us investigate the convective-boundary layer region of the trail $W^1 = \{e \ll \theta \ll e^{1/2},\ e \ll r - 1 \ll e^{-1}\}$ (the expression within the curly brackets indicates the order of the characteristic dimensions of the region in question). The right-hand side of (1.1) is inessential within this region, consequently the concentration here depends on the stream function only. The specific expression for the distribution of concentration in W^1 is obtained by matching with the solution in the diffusive boundary layer c^d obtained in [1]

$$c^1(z) = c^d(z, \theta)\big|_{\theta\to 0} = \operatorname{erf}\left(\sqrt{3/8(\beta+1)}\, z\right), \quad z = e^{-1}\psi \tag{2.1}$$

$$c^d(z, \theta) = \operatorname{erf}\left(\frac{z}{2\sqrt{t(\theta)}}\right), \quad t(\theta) = \frac{1}{2(\beta+1)}\left\{\frac{2}{3} + \cos\theta - \frac{\cos^3\theta}{3}\right\}$$

In this region the transfer of the substance arriving from the diffusion boundary layer takes place without any change along the streamlines.

To investigate the inner region of the trail $W^2 = \{e \ll r - 1 \ll e^{-1},\ \theta \ll e\}$ and the region of mixing $W^4 = \{e^{-1} \ll r,\ \psi \ll e\}$, in which the radial transfer is inessential, we write the equation of convective diffusion in the variables r, ψ, taking into account the fact that the first term within the brackets in the right-hand side of (1.1) can be neglected in these regions

$$\frac{\psi_\theta}{\sin\theta}\frac{\partial c}{\partial r} = e^2\left\{\psi_\theta{}^2\frac{\partial c^2}{\partial\psi^2} + (\psi_{\theta\theta} + \operatorname{ctg}\theta\psi_\theta)\frac{\partial c}{\partial\psi}\right\} \tag{2.2}$$

Here all coefficients must be expressed in terms of r and ψ and the expression for ψ given in (1.1) is utilized.

We shall consider the region of the rear stagnation point $W^3 = \{\theta \leqslant e,\ y = r - 1 \leqslant e\}$, in which the radial and tangential transfer are both important, together with the inner region of the trail W^2

The equation and the boundary conditions for W^3 written in the variables $Y = (r - 1)\, e^{-1}$, $S = e^{-1}\theta$ are as follows:

$$Y\frac{x^3}{\partial Y} - \frac{1}{2}S\frac{z^3}{\partial S} = (\beta + 1)\left\{\frac{\partial^2 c^3}{\partial Y^2} + \frac{\partial^2 c^3}{\partial S^2} + \frac{1}{S}\frac{\partial c^3}{\partial S}\right\} \tag{2.3}$$

$$c^3\big|_{Y=0} = 0, \quad \left.\frac{\partial c^3}{\partial S}\right|_{S=0} = 0$$

$$c^3\big|_{S\to\infty} = c^d\big|_{\theta\to 0} \to e^2\sqrt{\frac{3}{8\pi(\beta+1)}}\,YS^2$$

Here the equation is obtained from (1.1), while the third boundary condition represents the condition of matching with the solution of the diffusive boundary layer c^d defined in (2.1).

The equation and the boundary conditions for W^2 written in the variables y, $\zeta = e^{-2}\psi$ become

$$\frac{\partial c^2}{\partial y} = 2\frac{\partial}{\partial\zeta}\zeta\frac{\partial c^2}{\partial\zeta} \tag{2.4}$$

$$\left(\sqrt{\zeta}\frac{\partial c^2}{\partial\zeta}\right)_{\zeta=0} = 0, \quad c^2\big|_{\zeta\to\infty} \to e\sqrt{\frac{3(\beta+1)}{2\pi}}\,\zeta$$

Here the equation is obtained from (2.2) and the second boundary condition is obtained by matching with the solution in the convective-boundary layer region of the trail (2.1). The relations (2.3) and (2.4) must be complemented by the condition of matching

$$c^3\,(Y \to \infty) = c^2\,(y \to 0) \tag{2.5}$$

We see from (2.3) and (2.4) that the concentration in the region W^3 is of the order of e^2, and in W^2 of the order of e. Therefore in order to fulfil (2.5), we must require that

$$c^2\,(y,\ \zeta\,(y,\ \theta))\big|_{y\to 0,\ \theta=\mathrm{const}} \to 0 \tag{2.6}$$

In accordance with (2.4), (2.6) and with (2.3), (2.5), the distributions of concentration in the corresponding regions W^2 and W^3 are, respectively,

$$c^2(y,\zeta) = e\sqrt{\frac{3(\beta+1)}{2\pi}}\,[\zeta + 2y] \tag{2.7}$$

$$c^3(Y, S) = e^2\sqrt{\frac{3}{8\pi(\beta+1)}}\,Y\,[S^2 + 4(\beta+1)] \tag{2.8}$$

Since the concentration within the inner region of the trail does not satisfy the boundary condition at infinity (1.2) when $y \to \infty$, we must consider the mixing region

$W^4 = \{e^{-1} \ll r, \psi \ll e\}$. Introducing the variables $\rho = er$, $z = e^{-1}\psi$, we obtain the following expressions for W^4 ·

$$\frac{\partial c^{(4)}}{\partial \rho} = 2\frac{\partial}{\partial z} z \frac{\partial c^{(4)}}{\partial z} \left(\sqrt{z}\frac{\partial c^{(4)}}{\partial z}\right)_{z=0} = 0, \quad c^{(4)}|_{z\to\infty} = 1 \tag{2.9}$$

$$c^{(4)}|_{\rho\to 0} = \left[c^{(1)}(z) - \sqrt{\frac{3(\beta+1)}{2\pi}}\, z + c^{(2)}(y,\zeta)\right]_{y\to\infty} = c^{(1)}(z) + 2\sqrt{\frac{3(\beta+1)}{2\pi}}\,\rho$$

Here the equation is obtained from (2.2) and the initial condition (as $\rho \to 0$) is found by matching with the solutions in the regions $W^{(1)}$ and $W^{(2)}$. The solution of the problem (2.9) has the form

$$c^{(4)}(z, \rho) = A(z, \rho) * \left[\operatorname{erf}\sqrt{{}^3/_8(\beta+1)}\, z - \sqrt{\frac{3(\beta+1)}{2\pi}}\, z\right] + \sqrt{\frac{3(\beta+1)}{2\pi}}\,(z + 2\rho) \tag{2.10}$$

$$A(z, \rho) * u(z) = \int_0^\infty \frac{1}{2\rho} \exp\left\{-\frac{z+z^*}{2\rho}\right\} I_0\{(z, z^*)^{1/2}\rho\}\, u(z^*)\, dz^*$$

Here $I_0(z)$ represents a modified Bessel function of the first kind.

Let us quote the formulas for the distribution of concentration (in the spherical coordinate system) in the regions W_i $(i = 1, 2, 3, 4)$ of the diffusive trail of the drop, which follow from the expressions (2.1), (2.6), (2.7) and (2.9)

$$c^1 = \operatorname{erf}(e^{-1}\sqrt{{}^3/_8(\beta+1)}\,\psi) \tag{2.11}$$

$$c^{2,3} = \sqrt{\frac{3(\beta+1)}{2\pi}}\,[2e(r-1) + e^{-1}\psi]$$

$$c^4(r\to\infty) \to 1 - \frac{K}{r}\exp\left(-\frac{r\theta^2}{4e^2}\right), \quad K = \text{const}$$

Here the relation $\psi = \psi(r, \theta)$ is given by (1.1), $c^{2,3}$ denotes the distribution of concentration in the regions W^2 and W^3 which, as (2.7) and (2.8) imply, can be written as a single formula, and for W^4 we give the asymptotic behavior of the concentration at large distances from the drop; K denotes the total diffusive flux at the drop surface.

Expressions for the distribution of the concentration and the local Sherwood number which are uniformly suitable in e (over the whole interval $0 \leqslant \theta \leqslant \pi$, $r - 1 < e$), have the form

$$c(r, \theta) = c^d(r, \theta) + (r-1)\, j_{\min} \tag{2.12}$$

$$j(\theta) = \frac{\partial c}{\partial r}\bigg|_{r=1} = j^d(\theta) + j_{\min}, \quad j_{\min} = e\,[6\pi^{-1}(\beta+1)]^{1/2}$$

Here c^d and $j^d(\theta)$ are the quantities calculated according to the diffusive boundary layer approximation.

We can see that the minimum value of the local Sherwood number increases with the increasing viscosity of the drop, and the stern part of the drop makes a contribution towards the total diffusive flux only in the third order approximation with respect to e.

Behind the drop and along the axis the concentration at first increases linearly with the increasing distance y from the rear stagnation point, and later tends exponentially to its value at infinity (2.11). For a rigid particle the initial law of growth is proportional to the square root of y [4]. The solution within the inner region of the diffusive trail of the drop is more depleted than the corresponding region in the case of a rigid particle [4], and the concentration is of the order of $P^{-1/3}(\beta+1)^{1/3}$.

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